THEORETICAL PERTURBATION COMPUTATION OF ELECTROMAGNETIC EIGENMODES OF HOLLOW TOROIDAL WAVEGUIDES

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INTRODUCTION

The propagation of electromagnetic waves in a loss free inhomogeneous hollow waveguide with circular cross section and uniform plane curvature of the longitudinal axis is considered. The exact solution of Maxwell's equations in toruslike waveguides cannot be given with the correct boundary conditions. In a torus with small curvature the field equations can however be solved by means of an analytical approximation method. Using the Rayleigh-Schrödinger perturbation theory, eigenvalues and eigenfunctions containing first order correction terms are derived for the full spectrum of all modes, including the degenerate ones. The curvature of the axis of the waveguide is considered as a disturbance of the straight circular cylinder, and the perturbed torus-field is expanded in eigenfunctions of the unperturbed problem. Complicated series expansions are obtained, which can however be represented in closed form by means of the residue theorem. The field distortion increases with decreasing radius of curvature. This behaviour is proved by graphical representations of the field distribution.

THE WAVE EQUATION

The following procedure allows computing of wave propagation effects in loss free hollow waveguides with local circular cross section and uniform curvature. The so-called local or quasi toroidal coordinate system is conform to the metallic boundaries and reduces in the case of infinitesimal curvature to the common circular cylinder coordinate system. Thus the straight circular cylinder is obtained as a limitting case of the curved structure. Figure 1 gives the relationship of the dimensionless local toroidal coordinates (ξ, φ, α) with the rectangular coordinates (x, y, z). Using the transformation $\rho = a\xi$ and $s = R\alpha$ with ρ as quasiradial length and s as longitudinal coordinate measured along the curved axis, one obtaines

$$x = R\,h \cos\alpha \qquad (1)$$

$$y = R\,h \sin\alpha \qquad (2)$$

$$z = a\,\xi \sin\varphi \qquad (3)$$

with the metric coefficient $h = 1-\delta\xi\cos\varphi$ and the inverse aspect ratio $\delta = a/R$, where a is the minor and R the major radius of the torus, respectively; φ is the poloidal and α the toroidal angle. The interior of the torus is described by values of $0 \le \xi \le 1$. Applying this coordinate system to Maxwell's equations

$$\begin{aligned} \frac{\partial E}{\xi\,\partial\varphi} + j\beta a\, E_\varphi &= -j\omega\mu\, ha\, H_\varrho \\ -j\beta a\, E_\varrho - \frac{\partial E}{\partial\xi} &= -j\omega\mu\, ha\, H_\varphi \\ \frac{h\,\partial(\xi E_\varphi)}{\partial\xi} - \frac{h\,\partial E_\varrho}{\partial\varphi} &= -j\omega\mu\,\xi a\, H \end{aligned} \qquad (4)$$

$$\begin{aligned} \frac{\partial H}{\xi\,\partial\varphi} + j\beta a\, H_\varphi &= j\omega\epsilon\, ha\, E_\varrho \\ -j\beta a\, H_\varrho - \frac{\partial H}{\partial\xi} &= j\omega\epsilon\, ha\, E_\varphi \\ \frac{h\,\partial(\xi H_\varphi)}{\partial\xi} - \frac{h\,\partial H_\varrho}{\partial\varphi} &= j\omega\epsilon\,\xi a\, E \end{aligned} \qquad (5)$$

for stationary fields in homogeneous, isotropic and source free media, where all field components get the exponential dependence

$$e^{j(\omega t \pm \beta s)} \quad , \qquad (6)$$

we derive, by elimination of the transverse fields E_ϱ, E_φ, H_ϱ and H_φ according to Cap and Deutsch (1), a scalar inhomogeneous Helmholtz equation for the longitudinal components

$$(\Delta + \lambda)\, F = \delta\, L\, F \qquad (7)$$

using a bicomplex field intensity F suggested by Silberstein (2) with $E = hE_s$ and $H = hH_s$

$$F = h\left(E_s + i\sqrt{\frac{\mu}{\epsilon}}\, H_s\right) \quad , \qquad (8)$$

which can easily be decomposed in its physically meaningful constituents E_s and H_s. Δ is the transversal Laplacian operator and $L = L_1 - iL_2$ an involved perturbation operator of first order with

$$\Delta = \frac{\partial}{\xi\,\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \frac{\partial^2}{\xi^2\,\partial\varphi^2} \qquad (9)$$

$$L_1 = \frac{1+\gamma^2}{h(1-\gamma^2)}\left(-\cos\varphi\frac{\partial}{\partial\xi} + \frac{\sin\varphi}{\xi}\frac{\partial}{\partial\varphi}\right) \qquad (10)$$

$$L_2 = \frac{-2\gamma}{h(1-\gamma^2)}\left(\sin\varphi\frac{\partial}{\partial\xi} + \frac{\cos\varphi}{\xi}\frac{\partial}{\partial\varphi}\right) \qquad (11)$$

using the dimensionless quantity γ, depending on the transversal coordinates ξ and φ

$$\gamma = \frac{\beta}{\omega\sqrt{\mu\epsilon}\; h(\xi,\varphi)} \quad . \qquad (12)$$

The parameter

$$\lambda = \omega^2\mu\epsilon\, a^2(1-\gamma^2) \qquad (13)$$

is related via γ to the so far unknown propagation constant β, which serves as an eigenvalue of equation (7), so proposed by Lewin at al (3). The i-complex plane, introduced in eq. (8), must strictly be separated from the j-complex plane, which is commonly used for a more elegant description of the time dependence ($\cos\omega t \rightarrow e^{j\omega t}$) using complex phasors. Since the perturbation operator L is i-complex, the longitudinal field components E_s and H_s are coupled. Thus in a toroidal waveguide there are no TE- or TM eigenmodes, in the strict sense, as in the straight circular cylinder. All modes are hybrid and get 6 field components, resulting from the nonseparability of the Helmholtz equation (7) in toroidal coordinates.

PERTURBATIONAL APPROACH

The basic idea to solve our inhomogeneous wave equation (7) is to understand the curvature as a disturbance of the hollow waveguide with straight axis. The eigenvalues and eigenfunctions in the torus must continuously result from the solutions of the homogeneous differential equation ($\delta = 0$) while increasing the disturbance ($\delta > 0$). Thus the perturbed eigenfunctions can be represented in a power series expansion refering to the inverse aspect ratio $\delta = a/R$. The expansion coefficients are linear combinations of the unperturbed eigenmodes of the straight circular cylinder. Only tori with weak curvature ($0 \leq \delta << 1$) are considered in this paper. So the wanted expansions may be truncated after the linear term δ. An excellent description of the here used Rayleigh-Schrödinger perturbation theory of fist order is given by Schrödinger himself (4).

The homogeneous solutions

In the straight circular cylinder our wave equation (7) is reduced to

$$\left(\Delta + \lambda^{(0)}\right) F^{(0)} = 0 \quad , \qquad (14)$$

because of vanishing $\delta = 0$. This eigenvalue problem has a simple solution. The well known TE_{mn}- and TM_{mn}- eigenmodes with 5 field components are obtained, which build a complete orthogonal set.

Perturbation theory of first order

With weak perturbation ($\delta << 1$) the nature of the field distribution of all non-degenerate eigenmodes is changed only very slightly. There substantially exist furthermore TM- and TE- modes with only weak excitation of the so far missing second longitudinal component. Thus hybrid modes with 6 field components are derived, which can be classified as quasi-E- (EH-) and quasi-H- (HE-) modes, according to Brambilla and Finzi (5). For the solution of eq. (7) a linear perturbation ansatz of first order for the bicomplex field function and the propagation constant is made

$$F_\nu = F_\nu^{(0)} + \delta F_\nu^{(1)} \qquad (15)$$

$$\beta_\nu = \beta_\nu^{(0)} + \delta \beta_\nu^{(1)} \quad , \qquad (16)$$

where we combine all double indices (mn) to one (ν). The perturbation term is expanded in unperturbed eigenfunctions with so far unknown expansion coefficients

$$F_\nu^{(1)} = \sum_\mu c_{\nu\mu} F_\mu^{(0)} \quad , \qquad (17)$$

according to Lileg et al (6), who investigated the closed toroidal resonator. Inserting all this in eq. (7), while neglecting all terms of second order in δ, we derive infinite perturbation series, which can be represented in closed form using the residue theorem. The resulting perturbed expressions cannot explicitly be shown in this short summary. For more information see reference (7). But we remark, that the propagation constants of all non-degenerate circular cylinder eigenmodes are not altered to the first order in the toroidal waveguide

$$\beta_\nu^{(1)} = 0 \quad . \qquad (18)$$

The curvature of the axis indeed modifies the field configuration, but every mode is propagating with the same phase and group velocity as in the straight unperturbed hollow waveguide.

The degenerate case. For degenerate unperturbed eigenmodes of the circular cylinder, i.e. TE_{0n}- and TM_{1n}- modes, the situation is different. The field distortion is no longer as weak as in the non-degenerate case. The TM_{1n}-mode with an antisymmetrical orientation relative to the plane of curvature ($\propto$ - $\sin\varphi$) strongly couples to the TE_{0n}-mode. Thus antisymmetrical TM_{1n}- modes and TE_{0n}- modes of the straight circular cylinder are no quasi-stable torus-modes. They change their propagation constant by an amount

$$\beta_n^{(1)} = \pm \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}\, j_{1n}} \quad , \qquad (19)$$

which has already been pointed out by Jouguet (8), with j_{1n} as the n-th zero of Bessel's function J_1. Thus the degeneracy is removed in first order. In addition one finds an oscillation of energy between the perturbed TM_{1n}-mode and the perturbed TE_{0n}-mode

$$F_n = \left[F_{E_{1n}} \sin(\delta\beta_n^{(1)} s) + F_{H_{0n}} \cos(\delta\beta_n^{(1)} s) \right] e^{j\left(\omega t \pm \beta_n^{(0)} s\right)} \quad . \qquad (20)$$

At certain bending angles α the energy is concentrated in the perturbed TM_{1n}-mode, which has 6 field components and otherwise in the perturbed TE_{0n}-mode, which has no longitudinal electrical component. For general angles α we get mixed states F_n.

THE TORUS-FIELD - RESULTS AND DISCUSSION

For a better physical understanding of wave propagation phenomena in toroidal waveguides we give some plots of the obtained field intensities compared with the field of the straight circular cylinder. For instance the most important hybrid mode pair F_1 is investigated, see eq. (20), with which the unwelcome mode conversion $H_{01} \rightarrow E_{11}$ in circular hollow waveguide transmission lines can be described.

Field concentration and energy shift

In cross sections, where one of the two oscillating modes vanishes (see eq. (20)), Figures 2 and 3 show the radial electrical or magnetical field intensity of the remaining mode in question. The plots are taken at a waveguide radius of $\xi = 0.8$ as a function of the poloidal angle φ. The dashed lines give the behaviour for $\delta = 0$; the solid ones for $\delta = 0.03$ and $\delta = 0.06$. We notice a field shift towards $\varphi = \pi$ or a concentration near $\varphi = \pi$, respectively, similar to the results of Marcuse (9) obtained for curved optical fibres. In addition a plot of the energy flux density, i.e. the longitudinal component of the Poynting vector

$$P_s = \frac{1}{2}\, Re\left\{\vec{E} \times \vec{H}^*\right\} \cdot \vec{e}_s \qquad (21)$$

along a line in the plane of curvature ($\varphi = \pi$ or 0, respectively), taken for s = 0 (see eq. (20)) and same δ-values as above, indicates also a considerable shift in energy transport away from the center of curvature towards the outer boundary of the waveguide (plotted with negative values of ξ) as the curvature increases (see Figure 4).

CONCLUSIONS

Starting from the straight circular cylinder, the influence of a uniformly curved longitudinal waveguide axis has been considered. Graphical representations of the field distribution show a continuous shift of the transported energy towards the outer boundary, away from the center of increasing curvature, and a poloidal concentration near $\varphi = \pi$. In a future work quasi-toroidal hollow waveguides with slightly changing cross section radius a(s) will be investigated as a generalization of the present configuration.

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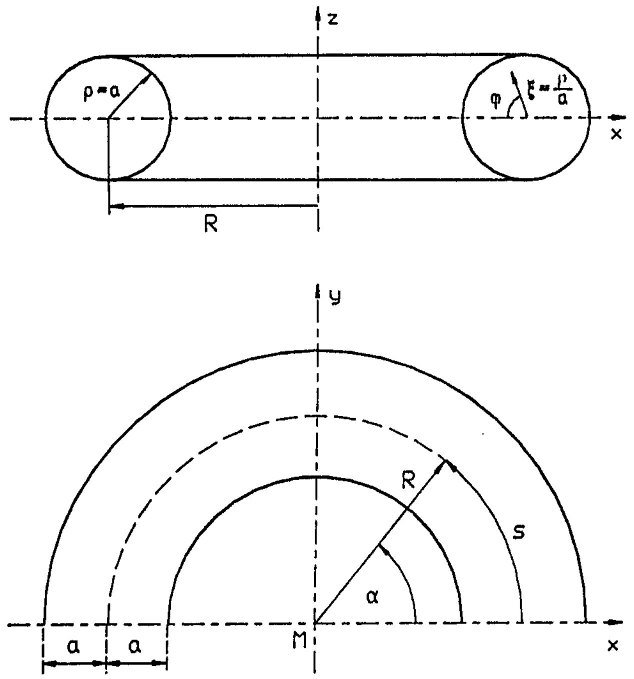

Figure 1 Torus with coordinate systems.

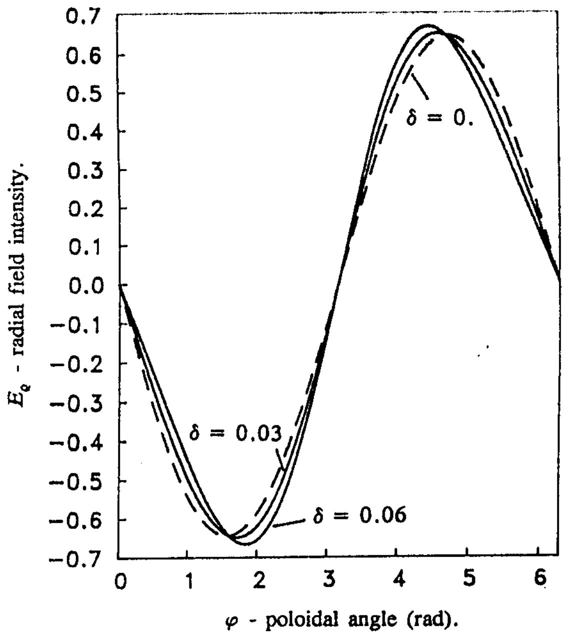

Figure 2 Angular field distribution of E_ϱ for the $F_{E_{1n}}$- torus-mode.

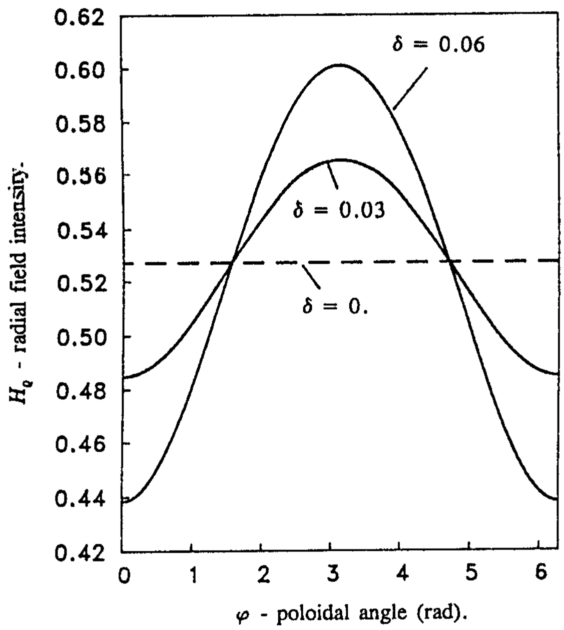

Figure 3 Angular field distribution of H_ϱ for the $F_{H_{0n}}$- torus-mode.

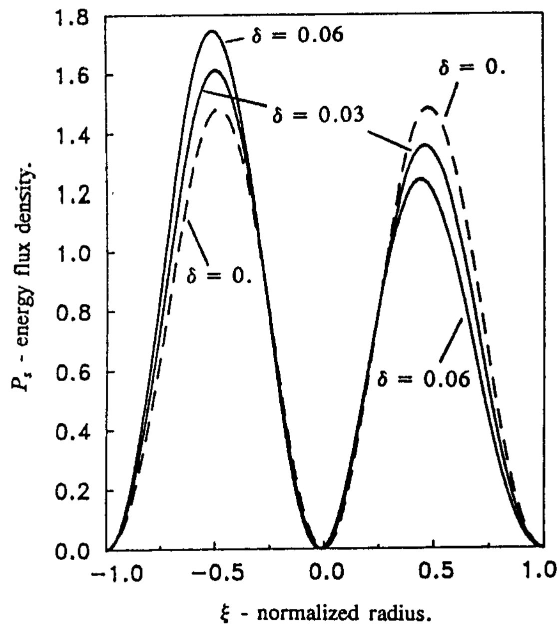

Figure 4 Energy transport along a line in the plane of curvature ($\varphi = \pi$ or 0) for the $F_{H_{0n}}$- torus-mode.