THEORETICAL PERTURBATION COMPUTATION OF ELECTROMAGNETIC EIGENMODES OF HOLLOW TOROIDAL WAVEGUIDES

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1. Introduction

The propagation of electromagnetic waves in a loss free inhomogeneous hollow waveguide with circular cross section and uniform plane curvature of the longitudinal axis is considered. The exact solution of Maxwell's equations in toruslike waveguides cannot be given with the correct boundary conditions. In a torus with small curvature the field equations can however be solved by means of an analytical approximation method. Using the Rayleigh-Schrödinger perturbation theory eigenvalues and eigenfunctions containing first order correction terms are derived for the full spectrum of all modes including the degenerate ones. The curvature of the axis of the waveguide is considered as a disturbance of the straight circular cylinder and the perturbed torus-field is expanded in eigenfunctions of the unperturbed problem. Complicated series expansions are obtained, which can however be represented in closed form by means of the residue theorem.

2. The Wave Equation

2.1 The local toroidal coordinate system (ξ, φ, α)

The following procedure allows computing of wave propagation effects in loss free hollow waveguides with local circular cross section and uniform curvature. The so-called local or quasi toroidal coordinate system is conform to the metallic boundaries and reduces in the case of infinitesimal curvature to the common circular cylinder coordinate system. Thus the straight circular cylinder is obtained as a limitting case of the curved structure. Figure 1 gives the relationship of the dimensionless local toroidal coordinates (ξ, φ, α) with the rectangular coordinates (x, y, z).

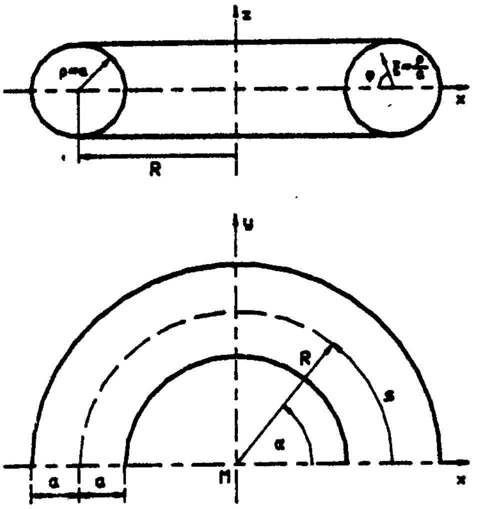

Figure 1. Torus with coordinate systems : rectangular coordinates (x, y, z) and local toroidal coordinates (ξ, φ, α) respectively (ϱ, φ, s) as generalized coordinates of the straight circular cylinder.

Using the transformation $\varrho = a\xi$ and $s = R\alpha$ with ϱ as quasiradial length and s as longitudinal coordinate measured along the curved axis one obtaines [1]

$$x = R\,h \cos\alpha \qquad y = R\,h \sin\alpha \qquad z = a\,\xi \sin\varphi \tag{1}$$

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with the metric coefficient $h = 1 - \delta\,\xi\cos\varphi$ and the inverse aspect ratio $\delta = a/R$, where a is the minor and R the major radius of the torus, respectively; φ is the poloidal and α the toroidal angle. The interior of the torus is decribed by values of $0 \le \xi \le 1$. After the definition of the coordinate system Maxwell's equations must be transformed for a better formal application of the usual perturbation theoretical scheme.

2.2 The field equations

In stationary fields in homogeneous, isotropic and source free media all field components get the exponential dependence

$$e^{j(\omega t \pm \beta s)} \quad , \tag{2}$$

which is omitted in the following computations. In addition only the negative sign in the exponent is used. It represents waves travelling in the positive s-direction. For the other direction of propagation β has to be substituted by $-\beta$. We use $H = hH_s$ and $E = hE_s$ as abbreviation for the longitudinal field components multiplied by the metric coefficient h. Then Maxwell's equations in local toroidal components

$$\begin{aligned}
\frac{\partial H}{\xi\,\partial\varphi} + j\beta a\, H_\varphi &= j\omega\epsilon\, ha\, E_\varrho \\
-j\beta a\, H_\varrho - \frac{\partial H}{\partial\xi} &= j\omega\epsilon\, ha\, E_\varphi \\
\frac{h\,\partial(\xi H_\varphi)}{\partial\xi} - \frac{h\,\partial H_\varrho}{\partial\varphi} &= j\omega\epsilon\, \xi a\, E \\
\frac{\partial E}{\xi\,\partial\varphi} + j\beta a\, E_\varphi &= -j\omega\mu\, ha\, H_\varrho \\
-j\beta a\, E_\varrho - \frac{\partial E}{\partial\xi} &= -j\omega\mu\, ha\, H_\varphi \\
\frac{h\,\partial(\xi E_\varphi)}{\partial\xi} - \frac{h\,\partial E_\varrho}{\partial\varphi} &= -j\omega\mu\, \xi a\, H
\end{aligned} \tag{3}$$

are transformed by elimination of the transverse fields E_ϱ, E_φ, H_ϱ and H_φ in a similar manner to [2], but where the notation is in some way inelegant, and so the resulting equations are here given in a much more compact and clearer operator description

$$\boxed{\begin{aligned}
\left(L^{(0)} + \lambda\right) E &= \delta\left(L_1^s\, E + L_2^s\, HZ\right) \\
\left(L^{(0)} + \lambda\right) HZ &= \delta\left(L_1^s\, HZ - L_2^s\, E\right) \quad .
\end{aligned}} \tag{4}$$

The equations (4) are the wanted coupled longitudinal equations, which perturbation theoretical treatment plays the central role in this paper. The differential operators are written in detail

$$L^{(0)} = \frac{\partial}{\xi\,\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \frac{\partial^2}{\xi^2\,\partial\varphi^2} \tag{5}$$

$$L_1^s = \frac{1+\gamma^2}{h\,(1-\gamma^2)}\left(-\cos\varphi\frac{\partial}{\partial\xi} + \frac{\sin\varphi}{\xi}\frac{\partial}{\partial\varphi}\right) \quad , \tag{6}$$

$$L_2^s = \frac{-2\gamma}{h\,(1-\gamma^2)}\left(\sin\varphi\frac{\partial}{\partial\xi} + \frac{\cos\varphi}{\xi}\frac{\partial}{\partial\varphi}\right) \quad , \tag{7}$$

where the dimensionless quantity $\gamma = \beta/(kh)$ and the metric coefficient $h = 1 - \delta\,\xi\cos\varphi$ are used. $k = \omega\sqrt{\mu\epsilon}$ is the wave number, $\delta = a/R$ the inverse aspect ratio, which serves as an expansion parameter in the following perturbation computation, and

$$\lambda = (ka)^2 (1 - \gamma^2) = (ka)^2 - \frac{(\beta a)^2}{h^2(\xi, \varphi)} \tag{8}$$

is the *coordinate dependent* eigenvalue of the differential equation system (4), which can be determined using the so far unknown propagation constant β [3]. E and H are the wanted transverse eigenfunctions and the wave impedance in free space $Z = \sqrt{\mu/\epsilon}$ is used for normalization. The perturbation operator L_1^s couples only modes of the same type (EE- or HH- coupling), and the perturbation operator L_2^s those of different type (EH- or HE- coupling). In the closed toroidal resonator there exists the particular case of toroidally uniform oscillations $(\partial/\partial\alpha = 0)$, where L_2^s vanishes in eq. (7) because of $\gamma = 0$. So the differential equation system (4) decouples, and E- or H- modes with only 3 field components are found. The resulting decoupled equations are identical to those published in [4]. Like the coupled system (4) shows, there in contrast only exist hybrid EH- (quasi-E) or HE- (quasi-H) modes in the hollow toroidal waveguide with generally 6 field components, which are derived in the following chapter. Indeed first of all the field equations (4) are transformed to a more compact appearance. By the help of a second complex plane the use of a bicomplex field intensity [5]

$$F = E + i\,ZH \tag{9}$$

enables a full decoupling of the field equations (4) and a scalar inhomogeneous wave equation

$$\left(L^{(0)} + \lambda\right) F = \delta\, L^s\, F \tag{10}$$

is obtained with the new perturbation operator $L^s = L_1^s - i\, L_2^s$. The so introduced i-complex plane must strictly be separated from the j-complex plane, which is commonly used for a more elegant description of the time dependence ($\cos \omega t \rightarrow e^{j\omega t}$) using complex phasors. In the following section the perturbation theoretical treatment of the inhomogeneous bicomplex wave equation (10) is studied in detail.

3. Theoretical Perturbation Computation of Eigenvalues and Eigenfunctions

The basic idea to solve our inhomogeneous wave equation (10) is to understand the curvature as a disturbance of the hollow waveguide with straight axis. The eigenvalues and eigenfunctions in the torus must continuously result from the solutions of the homogeneous differential equation $(\delta = 0)$ while increasing the disturbance $(\delta > 0)$. Thus the perturbed eigenfunctions can be represented in a power series expansion refering to the inverse aspect ratio $\delta = a/R$. The expansion coefficients are linear combinations of the unperturbed eigenmodes of the straight circular cylinder. Only tori with weak curvature $(0 \leq \delta \ll 1)$ are considered in this paper. So the wanted expansions may be truncated after the linear term δ. An excellent description of the here used Rayleigh-Schrödinger perturbation theory of first order is given in [6].

3.1 The homogeneous wave equation

In the straight circular cylinder our wave equation (10) is reduced to

$$\left(L^{(0)} + \lambda^{(0)}\right) F^{(0)} = 0 \tag{11}$$

because of vanishing $\delta = 0$. This eigenvalue problem has a simple solution. The well known E_{mn}- and H_{mn}- eigenmodes with 5 field components are obtained, which build a complete orthogonal set. Combining all double indices to one eq. (11) delivers with the following normalization integral (12) using the Kronecker-δ, where $F_\mu^{(0)}$ must be conjugated in the i-complex plane,

$$\int\limits_{\xi=0}^{1} \int\limits_{\varphi=0}^{2\pi} F_\nu^{(0)}\, F_\mu^{(0)^*} d\varphi\; \xi\; d\xi = \delta_{\nu\mu} \tag{12}$$

and with the boundary conditions

$$
\begin{aligned}
F_\nu^{(0)} &= 0 \quad \text{for } E_{mn} - \text{modes} \\
\frac{\partial F_\nu^{(0)}}{\partial \xi} &= 0 \quad \text{for } H_{mn} - \text{modes}
\end{aligned}
\tag{13}
$$

the unperturbed eigenfunctions

$$
F_\nu^{(0)} = \frac{1}{N_\nu} J_m(\tau_\nu \xi)\ \Phi_m(\varphi) \tag{14}
$$

with their eigenvalues $\lambda_\nu^{(0)} = \tau_\nu^2$. In doing so the transverse eigenvalue is found as a zero of the Besselfunction of the first kind or its derivative

$$
\tau_\nu = \left\{ \begin{array}{ll} j_{mn}, & \text{for } E_{mn} - \text{modes} \\ j_{mn}' & \text{for } H_{mn} - \text{modes} \end{array} \right\} \tag{15}
$$

with $m = 0, 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$. The normalization constant is

$$
N_\nu = \left\{ \begin{array}{ll} J_m'(j_{mn}) \sqrt{\frac{\pi}{2}(1 + \delta_{m0})} & \text{for } E_{mn} - \text{modes} \\ -i\, J_m(j_{mn}') \sqrt{\frac{\pi}{2}\left(1 - \frac{m^2}{j_{mn}'^2}\right)(1 + \delta_{m0})} & \text{for } H_{mn} - \text{modes} \ . \end{array} \right. \tag{16}
$$

The azimuthal ($\triangleq$ poloidal) field dependence is described by trigonometric functions

$$
\Phi_m(\varphi) = \left\{ \begin{array}{l} \cos m\varphi \\ \sin m\varphi \end{array} \right\} \ . \tag{17}
$$

Possible degeneracies in eigenvalue problems, if different eigenfunctions have the same eigenvalue, will be taken into account. But first the perturbation terms of all non-degenerate eigenmodes will be computed.

3.2 Non-degenerate Rayleigh-Schrödinger perturbation theory of first order

Every unperturbed eigenmode with 5 field components is continuously transformed to a torus-mode with 6 field components and the same transverse symmetry by the perturbation (curvature of the axis). These new toroidal eigenmodes can unfortunately not be represented in closed form but they can be approximated by a series expansion. With weak perturbation ($\delta \ll 1$) the nature of the field distribution of all non-degenerate eigenmodes is changed only very slightly. There substantially exist furthermore TM- and TE- modes with only weak excitation of the so far missing second longitudinal component. The mode designation of the toroidal modes can therefore be chosen in an analogous way to the straight circular cylinder [7]; thus hybrid modes with 6 field components are derived, which can be classified as quasi-E- (EH-) and quasi-H- (HE-) modes [2].

For the solution of eq. (10) a linear perturbation ansatz for the bicomplex field function and the square of the normalized propagation constant with $\lambda_\nu = (ka)^2 - \eta_\nu/h^2$ (see eq. (8)) and the abbreviation

$$
\eta_\nu = (\beta_\nu a)^2 \tag{18}
$$

is made

$$
\begin{aligned}
F_\nu &= F_\nu^{(0)} + \delta F_\nu^{(1)} \\
\eta_\nu &= \eta_\nu^{(0)} + \delta\, \eta_\nu^{(1)}
\end{aligned}
\tag{19}
$$

with $F_\nu^{(0)}$ like in eq. (14) as E_{mn}- or H_{mn}- mode of the straight circular cylinder and $\eta_\nu^{(0)} = (ka)^2 - \tau_\nu^2$. With

$$\beta_\nu = \beta_\nu^{(0)} + \delta\,\beta_\nu^{(1)} \tag{20}$$

and from the eqs. (18) and (19) the perturbation of the propagation constant is easily found

$$\beta_\nu^{(1)} a = \frac{\eta_\nu^{(1)}}{2\sqrt{\eta_\nu^{(0)}}} \quad . \tag{21}$$

The correction term $F_\nu^{(1)}$ and the change of the propagation constant $\eta_\nu^{(1)}$ will be derived. After substituting eq. (19) into the inhomogeneous wave equation (10) one obtaines a differential equation for the determination of the perturbed fields

$$L^{(0)}F_\nu^{(1)} - \hat{L}^s F_\nu^{(0)} = -\tau_\nu^2 F_\nu^{(1)} + \eta_\nu^{(1)} F_\nu^{(0)} \tag{22}$$

with the abbreviation

$$\hat{L}^s = L^s + 2\eta_\nu^{(0)}\xi\cos\varphi \tag{23}$$

while neglecting all terms of second order in δ and using the homogeneous wave equation (11). By the help of an expansion of the perturbation term $F_\nu^{(1)}$ in unperturbed eigenfunctions [6] with so far unknown expansion coefficients $c_{\nu\mu}$

$$F_\nu^{(1)} = \sum_\mu c_{\nu\mu} F_\mu^{(0)} \tag{24}$$

and the orthonormalization integral (12) the following solution is derived after some short transformations

$$\eta_\nu = \eta_\nu^{(0)} - \delta\,\hat{W}_{\nu\nu}$$
$$F_\nu = F_\nu^{(0)} + \delta \sum_{\mu\neq\nu} \frac{\hat{W}_{\mu\nu}}{\tau_\nu^2 - \tau_\mu^2} F_\mu^{(0)} \tag{25}$$

with the coupling integrals in form of inner products

$$\hat{W}_{\mu\nu} = \left(F_\mu^{(0)}, \hat{L}^s F_\nu^{(0)}\right) = \int_{\xi=0}^{1}\int_{\varphi=0}^{2\pi} F_\mu^{(0)*}\,\hat{L}^s\,F_\nu^{(0)} d\varphi\,\xi\,d\xi \quad . \tag{26}$$

$\hat{W}_{\mu\nu}$ cannot explicitly be specified in this short summary; one finds however, that because of the φ-integration in eq. (26) it is only necessary to perform the summation in the perturbation series in eq. (25) with such $\mu \triangleq pq$, for which the relation $p = m \pm 1$ holds. This is correct for the computation of the perturbation in first order $O(\delta)$ refering to the eigenfunction F_ν with $\nu \triangleq mn$. The summation over all $\mu \neq \nu$, which is indeed a double sum over all p and q with $\tau_{pq} \neq \tau_{mn}$, can so be reduced to a single sum over all $q = 1, \ldots, \infty$ with fixed p. For that reason one obtaines

$$\hat{W}_{\nu\nu} = 0 \quad , \tag{27}$$

that means the propagation constants of all non-degenerate circular cylinder eigenmodes are not altered to the first order in the toroidal waveguide. The curvature of the axis indeed modifies the field configuration but every mode is propagating with the same phase and group velocity as in the straight unperturbed hollow waveguide.

3.3 Degenerate Rayleigh-Schrödinger perturbation theory of first order

The only new idea of the degenerate perturbation theory is to find those linear combinations of the unperturbed degenerate eigenfunctions, which continuously result from the perturbed eigenfunctions with decreasing perturbation. There exists an infinite set of two by two degenerate eigenmodes. For $n = 1, \ldots, \infty$ one has with $\tau_n = j_{0n}' = j_{1n}$ the following degenerate pairs of eigenfunctions

$$F^{(0)}_{H_{0n}} = \frac{i}{\sqrt{\pi}\, J_0(\tau_n)} J_0(\xi\tau_n)$$
$$F^{(0)}_{E_{1n}} = \frac{\sqrt{2}}{\sqrt{\pi}\, J_1'(\tau_n)} J_1(\xi\tau_n) \left\{ \begin{matrix} -\sin\varphi \\ \cos\varphi \end{matrix} \right\} \quad . \tag{28}$$

Relative to the plane of curvature of the torus (see Figure 1) we will call the symmetrical field function ($\propto \cos\varphi$) the E_{1n}'-mode and the antisymmetrical ($\propto -\sin\varphi$) the E_{1n}''-mode, respectively. For a mode couple with fixed index n we make the ansatz of a so far unknown orthogonal substitution [6], where the modal index n is omitted furthermore for a clearer notation

$$F^{(0)} = b_1 F^{(0)}_{H_{0n}} + b_2 F^{(0)}_{E_{1n}} \quad , \tag{29}$$

for which we use exactly the same perturbation ansatz as in the non-degenerate case (19). After laborious computations the perturbation of the propagation constant has been found using $\beta^{(0)}a = \sqrt{(ka)^2 - \tau_n^2}$

$$\beta a = \left\{ \begin{matrix} \beta^{(0)}a \pm \delta \dfrac{ka}{\sqrt{2}\,\tau_n} & \textit{for the } E_{1n}'' - \textit{mode} \\ \beta^{(0)}a & \textit{for the } E_{1n}' - \textit{mode} \end{matrix} \right. \quad . \tag{30}$$

The perturbation theory shows, that a hybrid modal ansatz like in eq. (29) has no physical sense for the E_{1n}'-mode. Thus the symmetrical E_{1n}'-mode is quasi-stable. It does not degenerately couple to the H_0 modes, and therefore the computations of the non-degenerate perturbation theory are still valid for it. That means the E_{1n}'-mode couples to first order to E_0, E_2' and H_2'' modes but not to H_0 modes (see also in [8]). For the antisymmetrical E_{1n}''-mode in contrast the formalism of the degenerate perturbation theory must be used. It strongly couples (even for the weakest curvature) to the H_{0n}-mode building a degenerate hybrid pair of modes. Since we get two possible signs for the perturbation of the propagation constant $\beta^{(1)}$, there exist also two solutions for the couple of coefficients b_1 and b_2 in the orthogonal substitution (29), a fact, which matches with the wanted *two* unperturbed eigenfunctions, which substitute our two hybrid wave pairs in the limitting case of vanishing curvature

$$F^{(0)}_{\pm} = \frac{1}{\sqrt{2}} \left(F^{(0)}_{H_{0n}} \pm F^{(0)}_{E_{1n}''} \right) \quad . \tag{31}$$

Summarizing this section we obtain with $F^{(0)}_{\mu}$ from eq. (14) the perturbed eigenfunctions of the n-th degenerate hybrid wave pair

$$F_{\pm} = F^{(0)}_{\pm} + \delta \sum_{\tau_\mu \neq \tau_n} \frac{\left(F^{(0)}_{\mu}, \hat{L}^s F^{(0)}_{\pm} \right)}{\tau_n^2 - \tau_\mu^2} F^{(0)}_{\mu} \quad . \tag{32}$$

The degenerate eigenfunctions of the new transformed basis (31) undergo, like eq. (30) shows, a 'level splitting' for non-vanishing perturbation; thus the degeneracy is removed. A closed form representation of the field intensities of the toroidal modes is given in the next section.

4. The Torus-Field

4.1 Series representations

Starting from the derived series expansions for the perturbation of the eigenfunctions, which are lengthy and cumbersome and cannot explicitly be shown in this paper, the method how to get compact and closed expressions for the torus-fields is pointed out in this section. The double sums over all $\mu \triangleq pq$ with $q = 1, \dots, \infty$ in the eqs. (25) and (32) can be transformed in finite single sums with only two terms for $p = m \pm 1$. The inner perturbation series over all q can generally take the following two forms

$$\begin{aligned} S_E &= \sum_{q=1}^{\infty} \frac{j_{pq}}{\left(\tau_\nu^2 - j_{pq}^2\right)^{\psi}} \frac{J_p(\xi j_{pq})}{J_p'(j_{pq})} \\ S_H &= \sum_{q=1}^{\infty} \frac{(j_{pq}')^{\chi}}{\left(\tau_\nu^2 - j_{pq}'^2\right)^{\psi} \left(j_{pq}'^2 - p^2\right)} \frac{J_p(\xi j_{pq}')}{J_p(j_{pq}')} \end{aligned} \qquad (33)$$

with τ_ν like in eq. (15) and the exponents $\psi = 2$ or 3 and $\chi = 2$ or 4. A representation in closed form of the series expansions (33) is found by means of the residue theorem and complex integration. The so reached closed form expressions for the pole series S_E and S_H from eq. (33) cannot explicitly be shown in this short paper. In the further computations they replace the perturbation series (25) and (32) in the field representations, which final form will be given in the next section.

4.2 Closed form expressions of first order

After splitting the bicomplex field function (9) in real and imaginary part of the i-complex plane the j-complex amplitudes of the physically relevant field intensities are obtained. For instance the explicit expressions of the longitudinal field components multiplied by the metric coefficient h are particularly shown for the most important hybrid mode pair $F_\pm$, see eq. (32), with which the unwelcome mode conversion $H_{01} \rightarrow E_{11}$ in circular hollow waveguide transmission lines can be described. The corresponding transversal fields can simply be deduced by some derivatives using the system of Maxwell's equations (3).

$$\begin{aligned} E_\pm = &\mp \frac{1}{N_n} J_1(\tau_n \xi) \sin(\varphi) \pm \\ &\pm \frac{\delta}{4\tau_n^2 N_n} \Big\{ (ka)^2 \, \xi \, J_1(\tau_n \xi) + \\ &+ (\beta_n^{(0)} a)^2 \left[\tau_n \, (1 - \xi^2) \, J_2(\tau_n \xi) + 3 \, \xi \, J_1(\tau_n \xi) \right] \Big\} \sin(2\varphi) \end{aligned} \qquad (34)$$

$$
\begin{aligned}
ZH_{\pm} = & \frac{1}{\sqrt{2}\,N_n} J_0(\tau_n \xi) - \\
& - \frac{\delta}{4\tau_n^2 N_n} \Big\{ (ka)^2 \sqrt{2}\; \xi\, J_1{}'(\tau_n \xi) \cos\varphi + \\
& \quad + (\beta_n^{(0)} a)^2 \sqrt{2} \left[\tau_n (1 - \xi^2)\, J_1(\tau_n \xi) + \xi\, J_1{}'(\tau_n \xi) \right] \cos\varphi \pm \\
& \quad \pm ka\, \beta_n^{(0)} a \left[2\, \xi\, J_1(\tau_n \xi) - \tau_n\, J_2(\tau_n \xi) \right] \cos 2\varphi \Big\}
\end{aligned}
\tag{35}
$$

with the normalization constant

$$
N_n = \sqrt{\pi}\, J_1{}'(\tau_n) \tag{36}
$$

and the normalized propagation constant

$$
\beta_n^{(0)} a = \sqrt{(ka)^2 - \tau_n^2} \quad ; \tag{37}
$$

For a more profound description of the here shown method we refer to [9].

5. References

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